PROBLEMS IN BROWNIAN MOTION

1. Gaussian processes - necessary conditions for continuous version. Let $I=[0,1]$. A *Gaussian process* on I is a collection of random variables $X=(X_t)_{t\in I}$ on a common probability space $(\Omega,\mathcal{F},\mathbf{P})$, such that $X_{t_1},\dots,X_{t_k}$ has a joint Normal distribution for any $k\geq 1$ and any $t_1,\dots,t_k\in I$. Associate to the process its *mean function* $m(t):=\mathbf{E}[X_t]$ and *covariance kernel* $K(t,s)=\mathbf{E}\left[(X_t-m(t))(X_s-m(s))\right]$. We say that X is a continuous Gaussian process or an L^2 Gaussian process etc., if for $\mathbf{P}$-a.e. ω, the function $t\to X_t(\omega)$ is continuous or $L^2(I)$ etc., respectively.

1. Let X be a Gaussian process on I with covariance $K(\cdot,\cdot)$. Then K is positive definite in the sense that $\det\left(K(t_i,t_j)\right)_{i,j\leq n}\geq 0$ for any $t_1,\dots,t_n\in I$.

2. Let $m:I\to\mathbb{R}$ and $K:I^2\to\mathbb{R}$. A Gaussian process with mean $m(\cdot)$ and $K(\cdot,\cdot)$ exists if $K(\cdot,\cdot)$ is positive definite. [**Hint:** Kolmogorov's consistency theorem]

3. If X is a continuous Gaussian process with mean m and covariance K, then m and K are continuous functions on I and I^2, respectively.

4. Let $m:I\to\mathbb{R}$ and $K:I^2\to\mathbb{R}$ be continuous. A continuous Gaussian process with mean m and covariance K exists if and only a continuous centered Gaussian process with covariance K exists.

2. Gaussian processes - sufficient conditions for continuous version. Let $K:I^2\to\mathbb{R}$ be continuous and positive definite throughout this section. Define $d(t,s):=\sqrt{K(t,t)+K(s,s)-2K(t,s)}$. Problem 5 asserts that d is a pseudo-metric on I. Assuming this, let $N(\varepsilon)$ be the minimum number of open balls of radius ε (in the pseudo-metric d) needed to cover I. Let $J_K:=\int_0^\infty\sqrt{\log N(\varepsilon)}d\varepsilon$ (possibly infinite). As $N(\varepsilon)=1$ for $\varepsilon>D:=\text{dia}_d(I)$, the upper limit of the integral is D and hence, the finiteness of J_K depends on the behaviour of $N(\varepsilon)$ for small ε.

For each k, fix a 2^{-k}-net $0=t_{k,1}<t_{k,2}<\dots<t_{k,N_k}=1$ for I in the metric d. Fix a Gaussian process $(X_t)_{t\in I}$ with zero mean and covariance K (exists by Problem 2). Define $S_k(t)=X(t_{k,j})$ for $t=t_{k,j}$ and linearly interpolate in between.

5. Show that d_K is a pseudo-metric on I. Further, $d_K(t,s)$ is continuous with respect to the usual topology on I. In particular, $D:=\text{dia}_d(I)<\infty$ and $N(\varepsilon)<\infty$ for $\varepsilon>0$. Further, $\varepsilon\to N(\varepsilon)$ is Borel measurable and hence J_K is well defined.

6. Assume that $D > 0$ in all problems up to Problem 9. Then, $N(\varepsilon) \geq \frac{1}{2} D \varepsilon^{-1}$. In particular, $N_k \geq D2^k$.

7. For $\chi \sim N(0,1)$ and any $t > 0$, $\mathbf{P}(|\chi| > t) \leq 2e^{-t^2/2}$.

8. If $\lambda > 1$ then $\mathbf{P}(\|S_k - S_{k+1}\|_{\sup} > 2^{-k+2}\sqrt{2\lambda \log(N_k N_{k+1})}) \leq (N_k N_{k+1})^{1-\lambda}$. Almost surely, $\|S_k - S_{k+1}\|_{\sup} \leq 2^{-k+2}\sqrt{2\lambda \log(N_k N_{k+1})}$ for all but finitely many k. [**Hint:** Fix $t \in [0,1]$ and consider i, j such that $t_{k,j} \leq t < t_{k,j+1}$ and $t_{k,i} \leq t < t_{k,i+1}$].

9. If $J_K < \infty$, then $\sum_k 2^{-k}\sqrt{\log(N_k N_{k+1})} < \infty$. Almost surely, S_k converges uniformly on I to a random continuous function.

10. [*Dudley's theorem*] Let $K : I^2 \to \mathbb{R}$ be continuous and positive definite. If $J_K < \infty$, then a continuous Gaussian process with zero mean and covariance K exists. [**Note:** The case $D = 0$ may be dealt with directly].

11. Let $K : I^2 \to \mathbb{R}$ be continuous and positive definite. If $d(t,s) \leq C|t-s|^\beta$ for all $t, s \in I$ and for some $C < \infty$ and $\beta > 0$, then a continuous Gaussian process with zero mean and covariance K exists.

12. A continuous Gaussian process on I with zero mean and covariance $K(t,s) = t \wedge s$ exists. This process is called *standard Brownian motion*.

13. Let W be standard Brownian motion. Then, (a) $W_0 = 0$ a.s., (b) for any $t_1 < t_2 < \ldots < t_n$, the random variables $W(t_1), W(t_2) - W(t_1), \ldots, W(t_n) - W(t_{n-1})$ are independent, (c) $W(t) - W(s) \sim N(0, s)$ for $s < t$.

Notes: Dudley's theorem (Problem 10) is valid as stated if I is replaced by an arbitrary compact space. The proof needs some modification, however. The Dudley condition $J_K < \infty$ is close to being necessary. In fact, *Fernique's theorem* asserts that if $d(t,s) = d(t-s)$ for $t, s \in I$ (increments are stationary), then $J_K = \infty$ implies that there does not exist a continuous Gaussian process with covariance K.

These theorems are way stronger than is needed for the construction of Brownian motion. Elementary constructions such as the one by Lévy and Cisielski are very attractive, but quite specific to the covariance structure of Brownian motion. Kolmogorov-Centsov is another standard way in which textbooks present the construction. It is much weaker (and historically much earlier) than Dudley's theorem.